THEORETICAL SCALING OF THE IDF CURVES

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ABSTRACT

Under the condition that temporal rainfall is a stationary multifractal process, we derive scaling properties of the IDF curves. We find that the IDF values have approximately a power law dependence on duration D and return period T, with exponents that differ from those obtained in previous studies. The scaling relations depend to some extent on the definition of the return period. Theoretical findings are validated through simulation.

1. INTRODUCTION

The intensity-duration-frequency (IDF) curves are standard tools of hydrologic risk analysis and design. They are defined as follows. Let I_D be the annual maximum rainfall intensity in a period of duration D at a given location and denote by $i_p(D)$ the value exceeded by I_D with probability p. The IDF curves are plots of $i_p(D)$ against D for selected return periods T = 1/p. It has been repeatedly observed that, over a wide range of durations, the distribution of I_D satisfies the simple-scaling relation $I_D \overset{d}{=} r^H I_{rD}$, with H typically in the range 0.6-0.8 (e.g. Burlando and Rosso, 1996; Menabde *et al.*, 1999). It follows that the IDF curves satisfy $i_p(D) = g(p)D^{-H}$, where g(p) is some function of p. A second observation is that, at least in some cases and for small p, $g(p) \propto p^{-\alpha}$ for some α.

The simple scaling dependence of the annual maximum intensity I_D on duration D may appear at odds with the fact that temporal rainfall is not itself a self-similar process. Rather, the average intensity in [t, t+D], $I'_D(t)$, may to a first approximation be considered multifractal; see for example Hubert *et al.* (1993), Lovejoy and Schertzer (1995), Olsson *et al.* (1993), Menabde *et al.* (1997), and Schmitt *et al.* (1998). Hence, a compelling theoretical problem is to reconcile the multifractal scaling of rainfall with the self-similarity of the IDF curves and derive the exponents H and α from characteristics of the temporal rainfall series. Interest in these issues is not just theoretical (linking the IDF curves to characteristics of the rainfall process and possibly to climatic and atmospheric conditions), but also practical (extrapolation of the IDF intensities to values of D and T beyond the range estimable from historical records, regionalization of the IDF curves, etc.).

Recently, Benjoudi *et al.* (1997, 1999) have derived H and α from the codimension function of the rainfall process. Here we make the same basic assumptions as Benjoudi *et al.*, but depart in some important ways from their analysis and reach different conclusions about the scaling of the IDF curves.

2. ALTERNATIVE DEFINITIONS OF THE RETURN PERIOD

The IDF curves and their scaling properties depend to some extent on the way the return period T is defined. Let T(D,i) be the return period of events with average rainfall intensity i over duration D. The definition used above, T = 1/p, corresponds to taking

$$T_1(D,i) = \frac{1}{P[I_D > i]} \quad (1)$$

where I_D is the annual maximum of $I'_D(t)$. Alternatively, one may relate the return period to the marginal exceedance probability $P[I'_D(t)>i]$ as

$$T_2(D,i) = \frac{D}{P[I'_D(t) > i]} \quad (2)$$

The reciprocal of $T_2(D,i)$ is the expected number of non-overlapping D intervals in one year when $I'_D > i$. Other definitions of T are possible, for example based on the distribution of peak above threshold or on the upcrossing rate for the intensity level i. The IDF curves for the j^{th} definition of the return period T are plots of $i_j(T_j,D)$, the rainfall intensity for duration D and return period T_j. We want to see how i_j varies with D and T_j. Our analysis is based on a large deviation property of multifractal measures, which is derived first.

3. A LARGE-DEVIATION PROPERTY OF MULTIFRACTAL MEASURES

Consider a multiplicative cascade that starts at level 0 with a uniform unit measure in the unit interval. Denote by b the multiplicity of the cascade (an integer greater than 1), by B the cascade generator, by r an integer power of b, and by ε_r the average measure density in a generic cascade tile of length 1/r. It is well known (e.g. Schertzer and Lovejoy, 1987; Gupta and Waymire, 1993) that, for any given γ,

$$P[\varepsilon_r > r^{\gamma}] \sim r^{-c(\gamma)} \quad (3)$$

where ~ denotes equality as $r \to \infty$ up to a factor that varies slowly (e.g. logarithmically) with r and $c(\gamma)$ is the Legendre transform of the moment scaling function $K(q) = \log_b[B^q]$. The function $c(\gamma)$ may be obtained parametrically from K(q), as

$$\gamma(q) = \frac{dK(q)}{dq}, \qquad c(q) = q\gamma(q) - K(q) \quad (4)$$

Technically, Eq. 4 holds for $\gamma \le \gamma^*$ where γ^* is the value of γ associated with the moment order $q^* > 1$ for which $K(q^*) = q^* - 1$. The form of $c(\gamma)$ for $\gamma > \gamma^*$ plays an important role in the IDF analysis of Benjoudi *et al.* (1997, 1999), but not here, since we find that what matters is the function $c(\gamma)$ in the range $\gamma \le \gamma^*$.

For the scaling analysis of the IDF curves, we need to extend Eq. 3 to obtain the probability $P[\varepsilon_r > ar^{\gamma}]$ for any given positive number a. By using $ar^{\gamma} = r^{\gamma + \log_r a}$, $\lim_{r\to\infty} \log_r(a) = 0$, and $c(\gamma + \delta) = c(\gamma) + q(\gamma)\delta$ for small δ, we conclude that

$$P[\varepsilon_r > ar^{\gamma}] = P[\varepsilon_r > r^{\gamma + \log_r a}] \sim a^{-q(\gamma)} r^{-c(\gamma)} \qquad (5)$$

where ~ denotes asymptotic equality as $r \to \infty$, up to a slowly varying function of r.

4. Scaling of the IDF Curves for $T = T_2$

Next we use the result in Eq. 5 to obtain the scaling properties of $i_2(T_2,D)$ for large T_2. Equation 5 applies to measure densities ε_r with unit expected value. In the case of rainfall, $m = E[I'_D]$ is not necessarily 1. Therefore, I'_D has a unit-mean cascade representation of the type I'_D = $m\,\varepsilon_{D_o/D}$, where D_0 is the outer limit of the scaling regime for rainfall. Substituting $r = D_o/D$ and $\varepsilon_r = \varepsilon_{D_o/D} = I'_D/m$ into Eq. 5, one obtains

$$P[I'_D > m\,a\left(\frac{D}{D_o}\right)^{-\gamma}] \sim a^{-q(\gamma)}\left(\frac{D}{D_o}\right)^{c(\gamma)} \qquad (6)$$

for $D << D_o$. To see how i_2 depends on D and T_2, we recall from Eq. 2 that $i_2(T_2,D)$ satisfies $P[I'_D > i_2] = D/T_2$; hence, to obtain an expression for i_2, one should make the right hand side of Eq. 6 equal to D/T_2. This happens for

$$\begin{cases} \gamma = \gamma_1 \ \text{ such that } c(\gamma_1) = 1 \\ a = \left(\dfrac{T_2}{D_o}\right)^{1/q_1} \end{cases} \qquad (7)$$

where $q_1 = q(\gamma_1)$. The parameters γ_1 and q_1 are obtained from the moment scaling function K(q) of the rainfall process as illustrated in Figure 1.

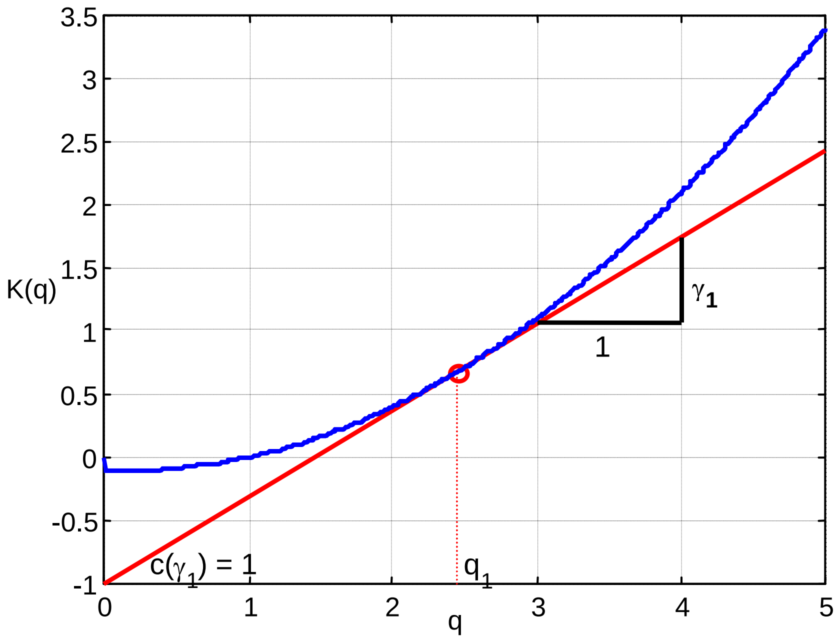

Figure 1 - Illustration of the IDF scaling parameters γ_1 and q_1.

From convexity of K(q) and the conditions K(1) = 0 and K(q*) = q* - 1, it follows that $\gamma_1 \leq 1 \leq \gamma^*$. With γ_1 and a in Eq. 7, Eq. 6 becomes

$$P[I'_D > m\left(\frac{T_2}{D_o}\right)^{1/q_1}\left(\frac{D}{D_o}\right)^{-\gamma_1}] \sim \frac{D}{T_2} \qquad (8)$$

Hence, for $D << D_o$, i_2 must scale with T_2 and D as

$$i_2 \propto T_2^{1/q_1} D^{-\gamma_1} \qquad (9)$$

This scaling result differs from the relation found by Benjoudi *et al.* (1997, 1999).

5. Scaling of the IDF Curves for T = T_1

To obtain the scaling properties of $i_1(T_1,D)$ with T_1 in Eq. 1, we make an assumption, which we have found to be accurate. Denote by $\overline{N}(p,D)$ the expected number of D intervals in one year when the average intensity I'_D

exceeds the (1-p) fractile of the annual maximum precipitation intensity, $i_p(D)$. We assume that, for fixed p, $\overline{N}(p,D) \equiv \overline{N}(p)$ independent of D. Support for this assumption comes from our own simulations of multifractal processes (see below) and the well-known fact that the IDF curves based on the return periods in Eqs. 1 and 2 are practically the same for $T_1 = T_2$ longer than about 10 years (Chow *et al.*, 1988).

Under this assumption, the intensities $i_1(T_1,D)$ are obtained by simply changing the return periods of the IDF curves $i_2(T_2,D)$, without modifying the curves themselves. In fact, by definition, the rainfall intensities $i_p(D)$ for different D and given p have all the same return period $T_1 = 1/p$. From Eq. 2 and the assumption above, the same intensities correspond to $T_2 = 1/\overline{N}(p)$ $= 1/\overline{N}(1/T_1)$. Substitution of the last relation into Eq. 9 gives

$$i_1(T_1,D) \propto \left[\overline{N}\left(\frac{1}{T_1}\right)\right]^{-1/q_1} D^{-\gamma_1} \qquad (10)$$

Since $\overline{N}(p)$ satisfies $\lim_{p\to 0} \frac{\overline{N}(p)}{p} = 1$, it is $\lim_{T_1\to\infty} \frac{T_2}{T_1} = 1$, meaning that for long return periods the IDF curves based on the definitions of T in Eqs. 1 and 2 are the same. As we have noted above, this is true in practice for return periods as short as 10 years or less.

6. NUMERICAL VALIDATION

To validate Eq. 9 and the assumption on $\overline{N}(p,D)$ that led to Eq. 10, we have performed numerical simulations using a locally multifractal model. The model considers the following key features of temporal rainfall: (i) the power spectrum of temporal rainfall has a breakpoint at about 2 weeks and is nearly flat above 2 weeks; (ii) in many climates, the events that dominate the IDF curves tend to come from a single season; and (iii) temporal rainfall displays an alternation of wet and dry periods.

Based on these features, we have formulated the following rainfall model. Each year is represented by a three-month long dominant season, which in turn is partitioned into 2-week intervals. Within each 2-week interval, the rainfall time series is an independent realization of a stationary multifractal measure of the beta-lognormal type, with moment scaling function $K(q) = C_\beta(q-1) + C_{LN}(q^2 - q)$, where C_β and C_{LN} are non-negative constants such that $C_\beta + C_{LN} < 1$. The associated IDF scaling exponents in Eqs. 9 and 10 are

$$\begin{cases} \gamma_1 = (C_\beta - C_{LN}) + 2\sqrt{(1-C_\beta)C_{LN}} \\ \dfrac{1}{q_1} = \sqrt{\dfrac{C_{LN}}{1-C_\beta}} \end{cases} \qquad (11)$$

For the validation of Eq. 9, we have simulated 400 years of rainfall at 2.3 seconds resolution using this model with $C_\beta = 0.1$ and $C_{LN} = 0.15$. We have then aggregated the data to 10 minutes before further analysis. An example simulation for a single three-month season is shown in Figure 2. For the chosen values of C_β and C_{LN}, Eq. 11 gives $\gamma_1 = 0.685$ and $1/q_1 = 1/2.45 = 0.408$.

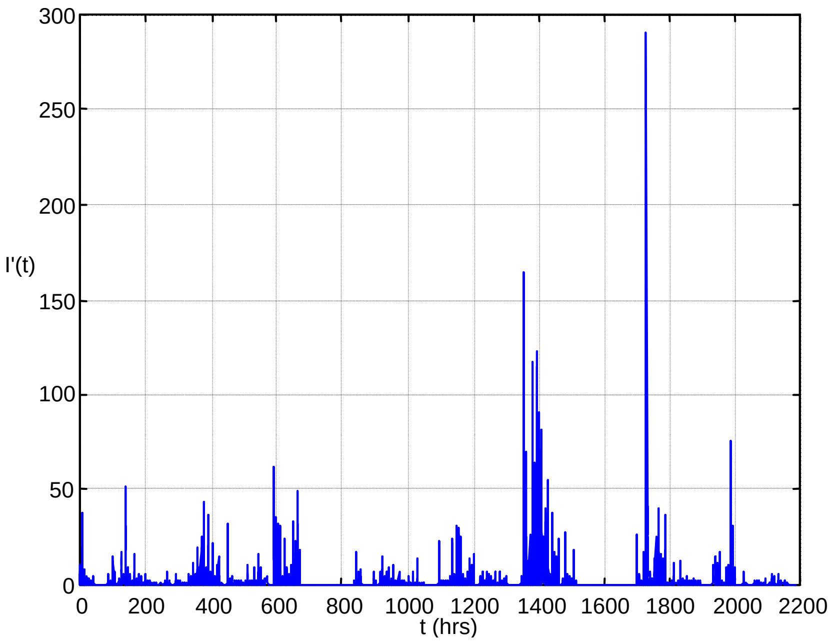

Figure 2 - Simulated 10-min rainfall for a three-month season.

Figure 3 shows the IDF curves extracted from the synthetic record, for return periods T_2 = 2, 5, 10, 20, 50, and 100 years. For each given return period, the intensity i_2 is observed to have a power dependence on duration D, with an exponent very close to the theoretical value -0.685. A significant finding is that the theoretical scaling is very accurately satisfied over a wide range of durations D, not just the infinitesimal durations under which Eq. 9 was derived.

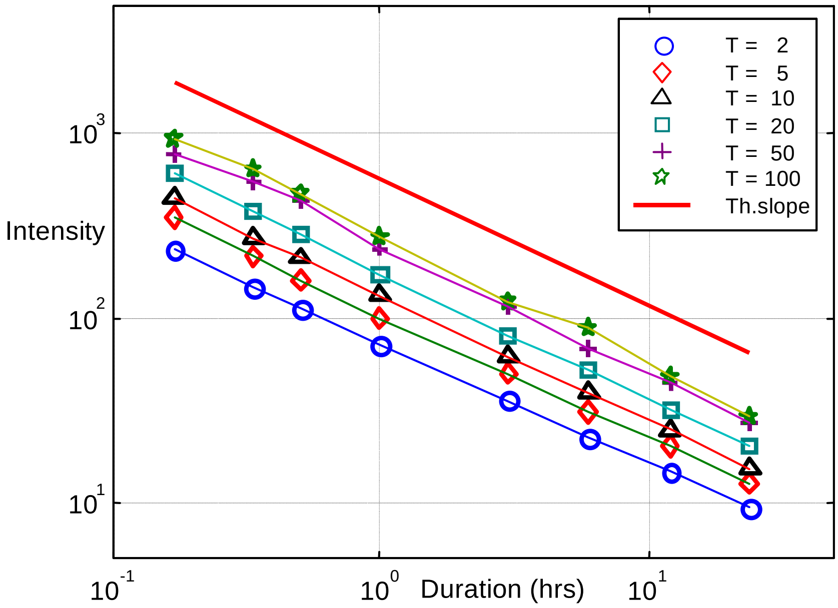

Figure 3 - IDF curves from the simulated rainfall sequence using $T = T_2$ in Eq. 2 (T in years). The thick solid line has the theoretical slope 0.685 and is shown for reference.

To validate the scaling of i_2 with T_2, we have calculated the geometric mean of the empirical rainfall intensities along each IDF curve in Figure 3 and plotted the resulting value against $\log(T_2)$. If Eq. 9 is correct, the plot should have a slope of $1/q_1 = 0.408$. Figure 4 shows that also this theoretical scaling relation is very closely satisfied.

Finally, we validate the assumption on $\overline{N}(p,D)$ by plotting the expected number of exceedances of $i_p(D)$ in a year as a function of duration D, for selected values of p. As Figure 5 shows, the assumption that $\overline{N}(p,D)$ is independent of D can be accurately made, at least for small values of p.

7. Conclusions

We have found that the IDF curves of stationary multifractal rainfall in

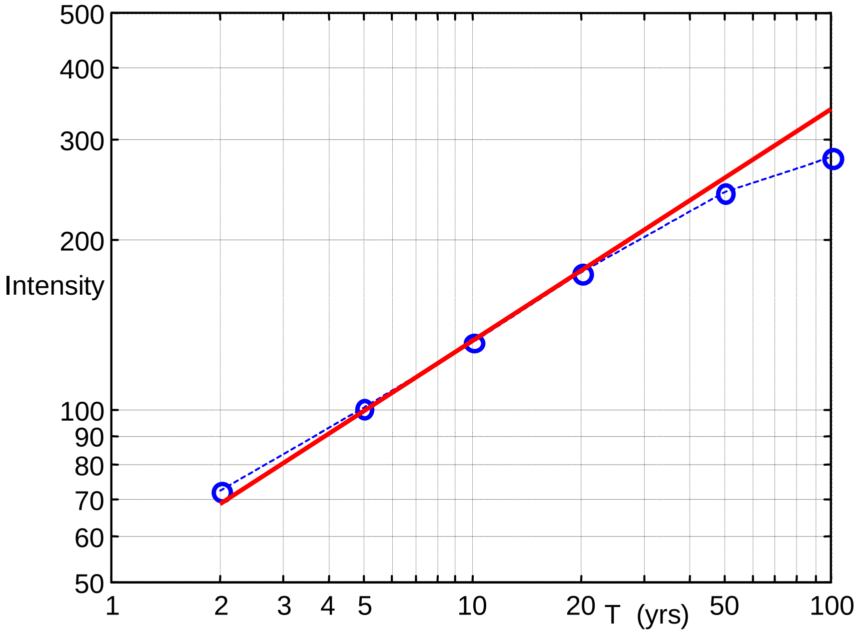

Figure 4 - Relationship between intensity and return period for $T = T_2$. The solid line has the theoretical slope 0.408. Each open circle is the geometric average of the empirical intensities in Figure 3 for a given value of T_2.

time are simple-scaling relative to duration D and, for large return periods T, have a power-law dependence on T. These qualitative results are in agreement with earlier findings by Benjoudi *et al.* (1997, 1999). However, the scaling exponents that we derive differ from the exponents in those earlier studies. Specifically, we find that the negative exponent of D is the value γ_1 of γ for which the codimension function of temporal rainfall, $c(\gamma)$, is 1 and that the exponent of T is $1/q_1$, where q_1 is the moment order associated with γ_1. The results have been validated through extensive numerical simulation.

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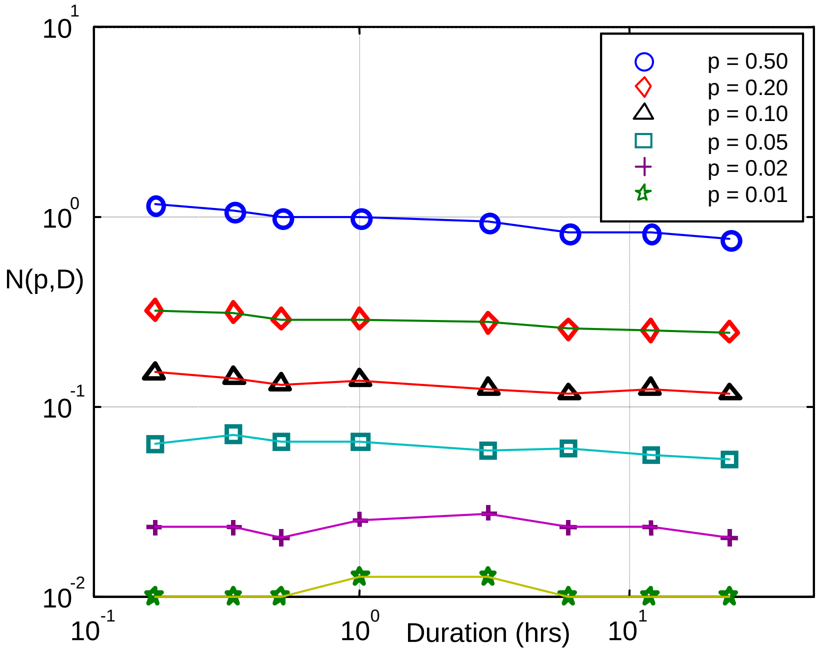

Figure 5 - Plots of $\overline{N}(p,D)$ against D for different exceedance probabilities p, using the synthetic rainfall record. $\overline{N}(p,D)$ is the expected number of intervals of duration D in one year when the average rainfall intensity I'_D exceeds the annual fractile $i_p(D)$.

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